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ABSTRACT

The problem of optimizing the thrust of a vertically ascending rocket is solved here under the assumption of isothermal atmosphere in two important cases: 1) the jet Mach number is sufficiently large; 2) the drag is a convex function of the velocity.

The first case embraces all physical drags and is valid for the Earth; the second extends to all atmospheres, but is restricted to drags that are fairly common.

With impulsive boosts in velocity admitted, the solution is shown to contain a finite number of such boosts in the sonic region of the rocket velocity, and to contain no coasting arcs except in the terminal stage.

An absolute minimum is proved with the aid of a Sufficient Condition applicable to problems of optimum control.

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1. INTRODUCTION

The problem of maximizing the summit altitude of a vertically ascending rocket, of which the Goddard Problem (1919) is a variant, has received considerable attention in the literature. One of the earliest attacks on the problem should be credited to Lewy (1944), instigated by Dr. R. H. Kent, then the Associate Director of the Ballistic Research Laboratories. Despite the notable advance achieved by Tsien and Evans (1951), numerous gaps in the theory still remain to be filled. As has been pointed out by Leitman, Ross, et al., the problem continues to be beset by the difficulty arising from the requirement that the mass be monotone. Solutions that meet this requirement have been obtained only in a few very special cases, typified by the work of Miele (1958), who treated flight in vacuum and the power law of drag.

In the present paper, which is an outgrowth of the author's unpublished work of 1949, reported at a Ballistic Research Laboratories Colloquium, the class of soluble cases is considerably broadened. With the assumption of isothermal atmosphere and the admissibility of infinite thrust, a solution is obtained in the following two cases:

1) The jet Mach number is sufficiently large.
2) The drag is a convex function of the velocity.

The first case is valid for the Earth; the second is restricted to a class of drags that are fairly common. The remaining case, where neither (1) nor (2) holds, is being left as a subject for future investigation.

A recapitulation of the relevant existing theory, designed to provide the necessary background for the current development, is incorporated in sections 2 and 5.

2. FORMULATION OF THE PROBLEM

The equation of motion of the rocket, subject to forces of gravity, drag, and thrust, is

$$\dot{m}c + m\dot{V} + \frac{1}{2} C_D(V,X)\rho(X)V^2 S + mg(X) = 0, \tag{1}$$

where m is the mass, V the velocity of the rocket, C_D the drag-coefficient, X the altitude, ρ the density of the air, S the cross-section, g the acceleration of gravity, c the jet velocity, and the superscript dot indicates the differentiation with respect to the time.

We shall introduce the simplifying assumptions:

$$\begin{aligned} &1)\quad C_D \text{ is a function of V only,} \\ &2)\quad g = \text{const.,} \\ &3)\quad \rho = \rho_o \exp(-X/\ell), \qquad \ell = \text{const.,} \end{aligned} \tag{2}$$

define the dimensionless parameters α, β by

$$\begin{aligned} &\alpha \equiv g\ell/c^2 \quad , \quad \beta \equiv 2\, m_o g/c^2 \rho_o S, \\ &0 < \alpha < \infty \quad , \quad 0 < \beta < \infty , \end{aligned} \tag{3}$$

where m_o is the initial mass, and dimensionless variables x, v, ω, y, and f by

$$\begin{aligned} &x \equiv gX/c^2 \quad , \quad v \equiv V/c \quad , \omega \equiv \log m_o/m \ , \\ &y \equiv \omega - v - x/\alpha \quad , \quad f \equiv C_D v^2 e^v/\beta \ . \end{aligned} \tag{4}$$

Then (1) becomes

$$\begin{aligned} &\phi \equiv -y' + fe^y/v + 1/v - 1/\alpha = 0 \ , \\ &x_o \leq x \leq x_1 \ , \end{aligned} \tag{5}$$

the prime indicating the differentiation with respect to x. The initial conditions are

$$x_o = 0 \ , \quad v(0) = 0 \ , \quad y(0) = 0; \tag{6}$$

the terminal conditions are not specified.

The quantities m and $\dot{m}$ in (1) are bounded by the inequalities

$$m \geq m_{min} \ , \quad 0 \leq -\dot{m} < \infty , \tag{7}$$

if infinite thrust is admitted as a mathematical convenience. Such a thrust, operating for an infinitesimal time, produces a finite positive jump Δ v, while y and (ω - v) remain continuous in virtue of (5) and (4). In terms of the new variables, (7) can be written

$$\begin{aligned} &\psi_1 \equiv \omega_{max} - y - v - x/\alpha \geq 0 \ , \\ &\psi_2 \equiv y' + v' + 1/\alpha \geq 0. \end{aligned} \tag{8}$$

The two unknown functions $y(x)$, $v(x)$ are connected by a differential constraint $\phi = 0$; the system therefore has one degree of freedom, which can be realized physically by a choice of an arbitrary $v(x)$, ideally regulated by a servo-mechanism controlling the flow rate $\dot{m}$. Functions $y(x)$, $v(x)$ will be admissible if y, v, y' satisfy the contraints (5) and (8) with the initial conditions (6), and if they are continuous except at corners, where y' and v may be discontinuous with $\Delta v \geq 0$. In the class of admissible functions we seek $v(x)$ that minimizes $-x_1$.

The problem is thus identified with the Problem of Mayer in the Calculus of Variations, complicated by the presence of algebraic and differential inequality constraints.

3. THE AUXILIARY PROBLEM

The differential constraint $\psi_2 \geq 0$, assuring the monotonicity of the mass, admits subarcs on which $\psi_2 = 0$ while $\psi_1 > 0$; i.e., the "burning" regime may be interrupted by the insertion of "coasting" subarcs. In order to avoid such complications let us consider an auxiliary problem characterized by the absence of the constraint $\psi_2 \geq 0$. While such a formulation, used by Tsien et al., automatically eliminates the aforesaid complication, it creates another one by admitting $\psi_2 < 0$ and $\Delta v < 0$. Of course, negative fuel consumption is a physical absurdity! The resulting solution would not be of physical interest, were it not for the curious fact that such an occurrence is precluded in certain practical cases. Indeed, if the constraints are satisfied anyway in the form $\psi_2 > 0$, $\Delta v \geq 0$, it is clear that the auxiliary and the actual problems have the same solutions. In particular, that such is the situation in both cases treated here will be shown in Theorems 1 and 2 of section 13. In terms of the quantities α and v, the two cases can be respectively characterized by:

1) α is sufficiently small;
2) $C_D v^2$ is convex.

Accordingly, we shall attack the Auxiliary Problem, which is in the standard form of the Problem of Optimum Control:

"We seek a function u(x) satisfying

$$\phi = -y' + g(x,y,u) = 0 \, , \qquad x_o \leq x \leq x_1 \, , \tag{9}$$

subject to the boundary conditions and the inequalities:

$$\begin{aligned} &x_o = a \, , \; y(x_o) = b, \\ &\Phi(x_1, \, y(x_1)) = 0 \, , \\ &\psi(x,y,u) \geq 0 \, , \end{aligned} \tag{10}$$

and minimizing some prescribed function

$$G(x_1,y(x_1)). \text{ "} \tag{11}$$

Here y, u, Φ, ψ are vectors of n, m, p, r components respectively, with $p < n + 1$. In our problem $n = m = r = 1$, $p = 0$; $G = -x_1$, $u = v$, and

$$\begin{aligned} &g \equiv (fe^y + 1)/v - 1/\alpha \, , \\ &\psi \equiv \omega_{max} - y - v - x/\alpha = \psi_1 \, . \end{aligned} \tag{12}$$

Since v' has disappeared from the problem, v has assumed the role of a "control" variable, which enters g(x,y,v) non-linearly. That the problem is non-singular is shown in section 8; the solution is obtained in sections 5 - 11 by the application of the Necessary Conditions I - IV and the Fundamental Sufficiency Condition of Weierstrass. The first one is the Multiplier Rule, comprising the Euler, the Transversality, and the Corner Conditions, treated respectively in sections 5, 6, and 7.

The existence and the character of the solution intimately depend on the nature of the drag coefficient $C_D(v)$, which is the subject of the next section.

4. SOME PROPERTIES OF THE DRAG

We shall assume the usual positiveness and the continuity of $C_D(v)$, the monotonicity of the drag,

$$\frac{d}{dv} (C_D v^2) > 0 \, , \tag{13}$$

and the asymptotic expansions

$$C_D = A_o + A_1 v + A_2 v^2 + \dots \qquad \text{as } v \to 0 \ ,$$

$$C_D = B_o + B_1/v + B_2/v^2 + \dots \quad \text{as } v \to \infty \ , \tag{14}$$

$$A_i \geq 0 \ , \quad B_i \geq 0 \ , \quad i = 0, 1, \dots \infty .$$

Then the logarithmic derivatives k and k' defined by

$$k \equiv d \log C_D / d \log v \ , \qquad k' \equiv dk/d \log v \tag{15}$$

have the properties:

$$\begin{aligned} & k(0) = k(\infty) = k'(0) = k'(\infty) = 0 \ , \\ & k(0+) > 0 \ , \quad k(\infty-) < 0 \ , \\ & k'(0+) > 0, \quad k'(\infty-) > 0 \ , \\ & \qquad k + 2 > 0 \ . \end{aligned} \tag{16}$$

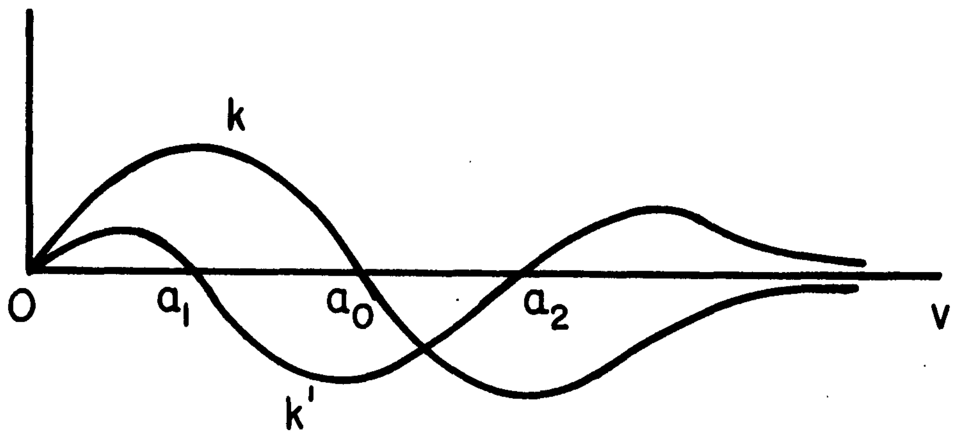

FIG. 1 LOGARITHMIC DERIVATIVES k (v) AND k'(v)

Furthermore, let $C_D(v)$ have a single maximum at, say a_o. Then k has a maximum at a_1, a zero at a_o, and a minimum at a_2, while k' has zeros at a_1 and a_2. It follows, in view of (16), that

$$(a_o - v)k > 0 \ , \quad (v - a_1)(v - a_2)k' > 0 \ , \tag{17}$$

$$0 < a_1 < a_o < a_2 < \infty \ .$$

In the analysis, the function f(v), defined in (4), and the derived functions H(v), h(v), defined herewith, will be extremely useful:

$$\begin{aligned} f &\equiv C_D(v)v^2 e^v/\beta > 0 \ , \\ H &\equiv vf_v - f \ , \\ h &\equiv H - \alpha f_v = (v - \alpha)\ f_v - f \ , \end{aligned} \tag{18}$$

with literal subscripts denoting the argument of differentiation.

In terms of k and k', these functions and their derivatives can be exhibited as follows:

$$\begin{aligned} f_v &= (f/v)\ (2 + v + k) > 0, \\ f_{vv} &= (f/v^2)\ \ [(2 + v + k)\ (1 + v + k) + v + k'] \ , \\ H &= f(1 + v + k) \ , \\ H_v &= vf_{vv} \ , \\ h &= f\ [(1 - \alpha/v)\ (2 + v + k) - 1] \ , \\ h_v &= (v - \alpha)f_{vv} \ . \end{aligned} \tag{19}$$

Special properties of these functions, obtained with the aid of (16), are tabulated on the following page:

$$f(0) = f_v(0) = H(0) = H_v(0) = h(0) = 0 \ ,$$

$$f(\infty) = f_v(\infty) = f_{vv}(\infty) = H(\infty) = h(\infty) = \infty \ ,$$

$$\frac{1}{2} f_{vv}(0) = C_D(0)/\beta > 0 \ , \tag{20}$$

$$h_v(0) = - \alpha \, f_{vv}(0) < 0 \ ,$$

$$2 \ f = f_{vv}(0)v^2 + \ldots . \quad \text{as } v \to 0 \ ,$$

$$2 \ H = f_{vv}(0)v^2 + \ldots . \quad \text{as } v \to 0 \ .$$

5. THE EULER EQUATIONS

Since the Lagrangian function of the Auxiliary Problem is $F = \lambda\phi + \mu\psi$, the extremals must satisfy the equations

$$
\begin{aligned}
& y' = g(x,y,u), \\
& \lambda' + \lambda g_y + \mu\psi_y = 0, \\
& \lambda g_u + \mu\psi_u = 0, \\
& \mu\psi = 0 \ , \ \psi \geq 0 \ ,
\end{aligned} \tag{21}
$$

where $\lambda(x)$, $\mu(x)$ are Lagrange multipliers. The subtitution from (12) into the Euler equations (21.2) and (21.3) now yields, in view of (18.2),

$$
\begin{aligned}
& \lambda' + \lambda f e^{y}/v - \mu = 0, \\
& (\lambda/v^2) \ (He^y - 1) - \mu = 0,
\end{aligned} \tag{22}
$$

leading to

$$\lambda = \lambda(0) \ \exp \int_0^x \left[(H - vf) \ e^y - 1 \right] dx/v^2 \ . \tag{23}$$

The use of the "switching function" $\mu(x)$ permits simultaneous consideration of subarcs lying in the region $\psi_1 > 0$, where $\mu = 0$, and of subarcs lying in the boundary $\psi_1 = 0$, where $\mu \neq 0$. Three regimes are distinguished, designated by I, B, and C:

I.	Impulsive thrust,	$\Delta v \neq 0$,
B.	"Burning",	$\psi_1 > 0$, $\mu = 0$,
C.	"Coasting",	$\psi_1 = 0$, $\mu \neq 0$.

An extremal is compounded of a B-subarc, with impulsive thrusts I occurring at a finite number of points, and a C-subarc appearing in the terminal stage only.

During the burning stage $\mu=0$, and the "optimality" condition (22.2) yields

$$e^y H(v) = 1 \ . \tag{24}$$

That a solution $v(y)$ of (24) exists follows from (20), which gives the range of H as $(0, \infty)$; that this solution is unique will be shown in section 9, with the aid of Condition II. Several conclusions can now be drawn. First, (8.1) implies $y < \infty$; then from (24) and (18) there follows

$$H > 0, \quad v \neq 0, \tag{25}$$

and therefore $v > 0$. Since the initial value $v\ (0) = 0$ violates the requirement (25), the burning stage must be preceded by an impulsive launching with a velocity v_o that satisfies (24) with the initial condition $y(0) = 0$; i.e.,

$$H(v_o) = 1 \ . \tag{26}$$

The initial discontinuity is thus specified by $v_-(0) = 0$, $v_+(0) = v_o$, and $\Delta\omega = \Delta v$. Of historical interest is the value of the gravity-drag ratio mg/D, which equals $1/fe^y$ in our symbols, and is optimized by (24) into H/f, or

$$mg/D = 1 + v + k \ . \tag{27}$$

The solution of the Euler equations is obtained by the differentiation of (24) with respect to x, yielding

$$- Hy' = H_v v' \quad , \tag{28}$$

followed by the substitution from (24) into (5), which in view of (18) now becomes

$$- \alpha\ Hy' = h \quad . \tag{29}$$

Then, from (28) and (29),

$$v' = h/\alpha\ H_v \quad , \tag{30}$$

and, provided $h \neq 0$, $v(x)$ is obtained by the inversion of the quadrature

$$x/\alpha = \int_{v_o}^{v} dH/h \tag{31}$$

$$= X(v) - X(v_o) \quad ,$$

where

$$\chi(v) = \int_1^v dH/h \tag{32}$$

defines a "rocket function" dependent only on the form of $C_D(v)$ and on the value of the parameter α. The equation of the extremal subarc now appears in the parametric form $x = x(v)$, $y = y(v)$, in consequence of (24) and (31). The special case $h = 0$ is solved in section 12.

During the coasting stage $\psi_1 = 0$, $\mu \neq 0$, and (5) becomes

$$vv' + f(v) \exp(-v - x/\alpha + \omega_{max}) + 1 = 0 \tag{33}$$

with the initial conditions corresponding to the "burnout", i.e. the solution of the equation $\psi_1 = 0$ with $y(v)$ and $x(v)$ furnished by (24) and (31).

It is to be noted that on the C-subarc, $\psi_1 = 0$, the number of degrees of freedom being zero, no variations are admitted. The implications are that the usual requirement $\mu \leq 0$ does not hold, and that the necessary and sufficient conditions are trivially satisfied.

6. THE TRANSVERSALITY CONDITION

In the control problem of section 3 the relation

$$\Big[(G_x + \lambda \cdot g)\, dx + (G_y - \lambda) \cdot dy\Big]_{x_1} = 0 \tag{34}$$

must hold for all dx and dy satisfying the differentiated equation $\Phi = 0$, the dot placed between vectors indicating their inner product. In the Auxiliary Problem $n = 1$, $G = -x_1$, and $p = 0$, so that dx and dy are arbitrary, and (34) reduces to

$$\begin{aligned} -1 + \lambda g &= 0 , \\ \lambda &= 0 \end{aligned} \tag{35}$$

at $x = x_1$. Three conclusions can be drawn. First, since both λ and μ cannot vanish simultaneously, $\mu(x_1) \neq 0$, so that $\psi_1(x_1) = 0$. Second, noting that $|g(x_1)| = \infty$ from (35), and recalling that $y < \infty$, $\alpha > 0$, $f < \infty$, we deduce

from (12) that $v(x_1) = 0$. Of course, both conclusions are physically obvious: x_1 must be reached with zero velocity after coasting with fuel consumed. The third conclusion,

$$\lambda(x_1 - 0) > 0 \tag{36}$$

follows from the observation that: 1) As $v \to 0$ the asymptotic value of g is $g \sim 1/v > 0$, in view of (12), (20), and (25), and that 2) $\lim (\lambda g) = 1$ as $x \to x_1$, in view of (35). Now, since $\lambda(x)$ cannot change its sign in virtue of (23), the inequality (36) implies

$$\lambda(x) > 0 \quad \text{for} \quad x_o \leq x < x_1 \ , \tag{37}$$

which requirement can be satisfied by choosing

$$\lambda(0) = 1 \ . \tag{38}$$

For future use, we note the following asymptotic values as $v \to 0$:

$$g \sim 1/v \ , \ \lambda \sim v \ , \ \lambda' \sim - 1/v \ , \ \mu \sim - 1/v \ , \tag{39}$$

which can now be obtained from (35), (22), and (20).

The existence and the continuity of the multipliers $\lambda(x)$, $\mu(x)$, required by the Multiplier Rule, is now assured between corners of a minimizing arc.

7. THE CORNER CONDITION

At a "free" corner, the relations

$$\Delta\ (\lambda . g) = 0 \quad , \quad \Delta\lambda = 0 \tag{40}$$

must hold, with Δ denoting a jump. Noting that in our problem, with $n = 1$, (40) implies $\Delta g = \Delta y' = 0$, and recalling that $\Delta y = 0$ in virtue of (5), we deduce from (24), (29), and (18) that

$$\Delta H = \Delta h = \Delta f_v = 0 \tag{41}$$

on a B-subarc. The definition of H now implies

$$\Delta f/\Delta v = f_v \ , \ \Delta f_v = 0 \ , \tag{42}$$

from which the transition values v_- and v_+ can be determined.

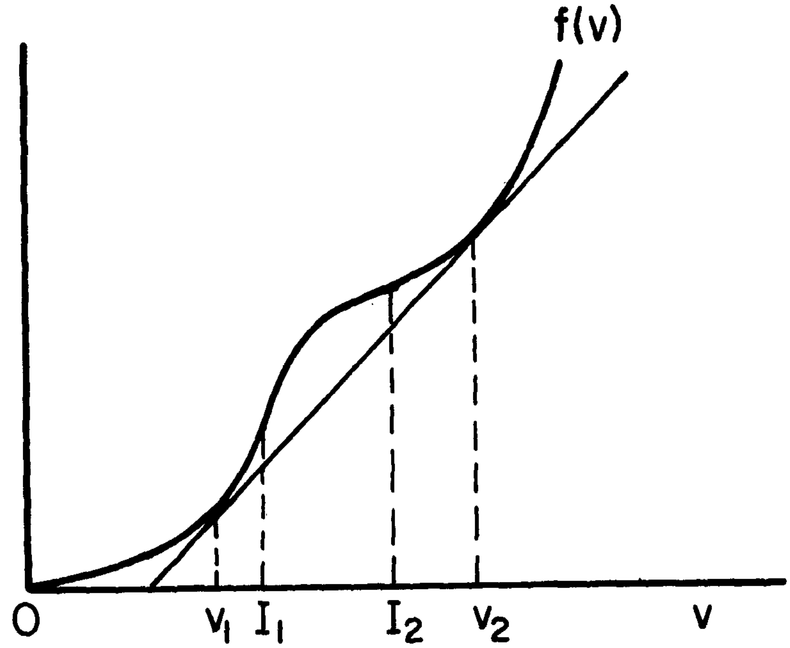

FIG. 2 DOUBLE TANGENT AND POINTS OF INFLECTION

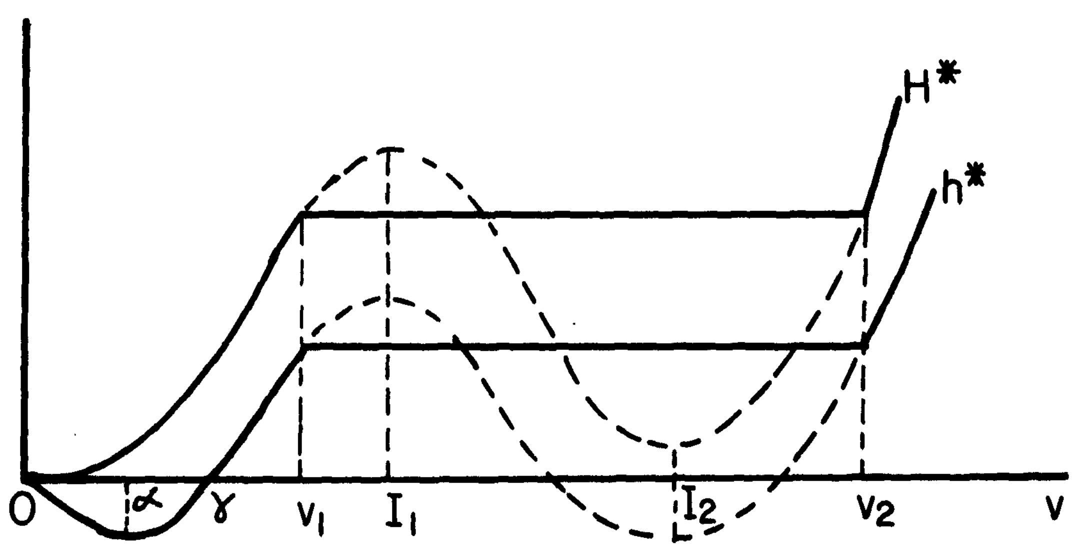

FIG. 3 H (v) , h (v) -----
$H^*(v)$, $h^*(v)$ ———

The transition values at the corner are then determined as follows:

$$\begin{aligned} v_- = v_1 \, , \; v_+ = v_2 \qquad & \text{if } v_1' > 0 \\ v_- = v_2 \, , \; v_+ = v_1 \qquad & \text{if } v_2' < 0 \, . \end{aligned} \tag{46}$$

It will be shown in section 9 that such a jump in velocity is required by the Weierstrass Condition II whenever the Corner Condition is satisfied.

The results of the last paragraphs can be easily generalized for any N, with (44) replaced by

$$f_{vv} \prod_{1}^{2N} (v - I_i) > 0 \, , \tag{47}$$

there being a velocity jump for each one of the N double tangents.

At the junction of the B and C-subarcs the Corner Condition is satisfied with

$$\Delta v = \Delta y = \Delta y' = \Delta\lambda = \Delta\mu = 0 \, , \quad \Delta v' < 0 \, . \tag{48}$$

8. THE HILBERT CONDITION

With $n = m = r = 1$, the four unknown functions $y(x)$, $\lambda(x)$, $u(x)$, $\mu(x)$ are related by the four equations (21). The highest derivatives being (y', λ', u, μ), the non-vanishing of the Jacobian determinant is the Hilbert Condition

$$\begin{vmatrix} F_{uu} & \psi_u \\ \mu\psi_u & \psi \end{vmatrix} \neq 0 \, , \tag{49}$$

or

$$|F_{uu}| \neq 0 \quad \text{if } \psi > 0 \, . \tag{50}$$

Here F is the Lagrangian function, and F_{uu} is generally an m x m matrix. The condition assures the existence of the highest derivatives listed above, as well as their piecewise continuity of class C^{k-2} if g and ψ are of class C^k, and is a direct consequence of the Legendre Condition III'.

In our problem (50) becomes

$$\lambda f_{vv}/vH \neq 0 \, , \tag{51}$$

and, since λ, v, and H are positive by (25) and (37),

$$f_{vv} \neq 0 \ . \tag{52}$$

Provided this requirement is met on the B-subarc, Hilbert Condition is satisfied, and since g is analytic in our problem, the functions (y,λ,v,μ) are analytic between corners. It is noteworthy that the use of the velocity v as a control variable, in place of the thrust $v\omega'$, removes the apparent singularity of the original problem.

9. CONDITIONS OF LEGENDRE AND WEIERSTRASS

The necessary conditions III and II, modified by the inclusion of the control variable u among the set of slope functions, can be written for the one dimensional case, n = m = 1, as

$$\begin{aligned} &[\lambda g_{uu}\ (x,\ y,\ u)\ +\ \mu\psi_{uu}]\quad \delta u^2 \geq 0, \\ &E \equiv \lambda\ [g(x,\ y,\ \bar{u})\ -\ g(x,\ y,\ u)] \geq 0, \end{aligned} \tag{53}$$

for all $(x,\ y,\ u,\ \lambda,\ \mu)$ belonging to the minimizing extremal, and for all $\bar{u} \neq u$ and satisfying $\phi = 0$. In our problem (53), with the aid of (12) and (24), reduces to

$$\begin{aligned} &\lambda f_{vv}/vH \geq 0\ , \\ &(\lambda/\bar{v}H)\ [\bar{f}\ -\ f\ -\ (\bar{v}-v)f_v]\ \geq 0, \end{aligned} \tag{54}$$

where $\bar{f} \equiv f(\bar{v})$, and finally, since v, H, and λ are positive, to the requirements that

$$\begin{aligned} &f_{vv} \geq 0\ , \\ &\bar{f}\ -\ f \geq (\bar{v}\ -\ v)f_v \end{aligned} \tag{55}$$

hold on every B-subarc.

In the language of geometry, (55) implies that v must be restricted to the domain where the tangent to the curve f(v) lies entirely below the curve. For drag of type 1, (55) is automatically satisfied; for drag of type 2, where (44) and (45) hold, (55) is equivalent to the requirement

$$(v - v_1)\ (v - v_2) \geq 0\ , \tag{56}$$

where v_1 and v_2 are the points of contact of the double tangent. The exclusion of the interval (v_1, v_2) from the B-subarc then demands that a corner occur when v reaches the values v_1 or v_2, as described in (46). Conversely, the occurrence of such a corner satisfies the requirement $E \geq 0$, the equality holding only at corners for $v = v_1$, $\overline{v} = v_2$, and conversely. Consequently, $E > 0$ except on a set of measure of zero, so that the Weierstrass Condition holds in its strengthened form II'.

A fortiori, the strengthened Legendre-Clebsch Condition III'; i.e.,

$$f_{vv} > 0 \tag{57}$$

also holds, from which two consequences follow. First, (57) establishes the Hilbert Condition (52); second, with the aid of (19.4) it implies that $H_v > 0$. We conclude that H(v) is monotonic in the domain defined by (56), and has an inverse H^{-1}, thus assuring the uniqueness of the solution v(y) of the equation (24). In view of this fact, it is convenient to replace H and h in all the equations referring to the B-subarc by H* and h* (see Fig. 3) defined by

$$\begin{aligned} &H^* \equiv H, \quad h \equiv h^* && \text{if } (v - v_1)\ (v - v_2) \geq 0\ , \\ &H^* \equiv H(v_1)\ ,\ h^* \equiv h\ (v_1) && \text{if } (v - v_1)\ (v - v_2) \leq 0\ . \end{aligned} \tag{58}$$

In the future, if no confusion results, the asterisks of H and h will be dropped.

10. THE JACOBI CONDITION

The second variation in the control problem defined by (9) - (11) with $n = m = r = 1$, $p = 0$ can be written

$$d^2J = \Big[(F_x - y'F_y)\ dx^2 + 2F_y dy dx + d^2G\Big]_{x_1} + \int_{x_0}^{x_1} (F_{yy}\ \delta y^2 + 2F_{yv}\delta y\delta v + F_{vv}\delta v^2)\ dx\ , \tag{59}$$

where F is defined by $F = \lambda\phi + \mu\psi$, $\phi = - y' + g(x, y, v)$, $\mu\psi = 0$, $\psi \geq 0$. The necessary Jacobi Condition is that

$$d^2 J \geq 0 \tag{60}$$

must hold on a minimizing extremal for all dx, dy, and for all δy, δv satisfying the differentiated equations $\phi = 0$, $\mu\psi = 0$.

Observe that $F_y = - \lambda'$ from (21.2), and that our problem possesses the following special features: 1) $G = - x_1$, so that $d^2G = 0$; 2) $F_x = - \mu/\alpha$ in view of (12); 3) u = v; and $F_{vv} = \lambda g_{vv}$, since ψ_1 is linear in v; 4) δy and δv must satisfy

$$\begin{aligned} - \delta y' + g_y \delta y + g_v \delta v &= 0 , \\ \mu \, (\delta y + \delta v) &= 0 \quad . \end{aligned} \tag{61}$$

Furthermore, on the B-subarc, $x_0 \leq x \leq \xi$, where ξ is the "burnout" point, observe that $\mu = 0$, $g_v = 0$ by (21.3), and the initial condition $y(0) = 0$ implies $\delta y(0) = 0$, and hence the solution of (61) is $\delta y \equiv 0$, δv arbitrary. On the other hand, on the C-subarc, $\xi \leq x \leq x_1$, $\mu \neq 0$, and the continuity of δy in (61) implies $\delta y(\xi + 0) = \delta y(\xi - 0) = 0$. Hence the solution of (61) is $\delta y = \delta v \equiv 0$, in agreement with the remark in the last paragraph of section 5. Now, since $\delta y = 0$ everywhere, $dy = y' dx$, and (59) becomes

$$d^2 J = \Big[(-\mu/\alpha - \lambda' g) \, dx^2\Big]_{x_1} + \int_{x_0}^{\xi} \lambda g_{vv} \delta v^2 \, . \tag{62}$$

Finally, at $x = x_1$ (39) yields $\mu \sim - 1/v$ and $\lambda' g \sim - 1/v^2$ as $v \to 0$; on the B-subarc $\lambda g_{vv} > 0$ by (53.1) and (57). Since α and v are positive, we conclude that the Jacobi Condition is satisfied in its strengthened form IV', $d^2 J > 0$.

An immediate consequence is that in our problem, with n = 1, IV' assures the existence of a field. Let a family of extremals $y(x,\theta)$ be defined by (9) - (11) with the initial condition (10.1) replaced by

$$x_0 = a \ , \ y(x_0) = b + \theta \ , \tag{63}$$

where θ is a family parameter. That the region bounded by $x = a$, $\Phi(x,y) = 0$ is, indeed, a field follows from the following considerations: 1) The extension of IV' to $\theta \neq 0$ is trivial; 2) IV' assures the simple covering of

the region; i.e., the existence of the function θ (x,y), and hence of the slope functions $u(x,y)$ and multipliers $\lambda(x,y)$, $\mu(x,y)$; 3) in a one-dimensional problem, the Euler equations suffice to assure that the Hilbert integral is independent of the path.

11. THE SUFFICIENCY CONDITION

We resort to the following variant of the Fundamental Sufficiency Condition of Weierstrass, proved in Appendix:

"Let a family $y(x,\theta)$ of extremals of a control problem be generated by the initial conditions (63), involving θ as a parameter. If this family constitutes a field, and if each extremal of the field satisfies I and II$'$ with the appropriate initial conditions, then the extremal for $\theta = 0$ yields an absolute minimum of the control problem."

Note that Conditions I and II$'$ have been established in sections 5-9 for $\theta = 0$, and that their extension to the family defined by (63) is trivial. Furthermore, the existence of a field has been proved in section 10. We conclude that the hypothesis of the theorem is satisfied, and that our extremal therefore yields an absolute minimum of the Auxiliary Problem.

12. THE STEADY STATES OF MOTION

Aside from their intrinsic interest, the lemmas of this section are required in the proof of the Basic Theorems of section 13.

Lemma 1: "The function $h^*(v)$ has one and only one positive zero, γ."

The proof proceeds from (18), (19), (20), (57), (58). Two cases are distinguished:

case 1. α is outside (v_1, v_2) (See Fig. 3)

Then the relations $h = (v-\alpha)\, f_v - f$, $h_v = (v-\alpha)\, f_{vv}$, $f_{vv} > 0$ imply that h^* has one and only one stationary point,

$$\min h^* = h(\alpha) = -\, f(\alpha) < 0. \tag{64}$$

Since the minimum is negative, the relations $h^*(0) = 0$, $h^*(\infty) = \infty$, and the continuity of h^* imply the conclusion of the lemma, with

$$\alpha < \gamma < v_1 \quad \text{or} \quad v_2 < \alpha < \gamma \ , \qquad (65)$$
$$(v - \gamma)h^* > 0 \ .$$

case 2. α is inside (v_1, v_2)

Then h^*, stationary on the interval (v_1, v_2), attains there a minimum, $\min h^* = h(v_1)$. That the minimum is again negative is implied by $h(0) = 0$, $h(\alpha) < 0$, and $0 < v_1 < \alpha$; finally, $h(v_1) = h(v_2) < 0$ and $h^*(\infty) = \infty$ imply the conclusion of the lemma, with

$$v_1 < \alpha < v_2 < \gamma \ , \qquad (66)$$
$$(v - \gamma)\ h^* > 0 \ .$$

Lemma 2: "On the burning subarc of an extremal, the acceleration of the rocket cannot change its sign."

The proof proceeds from (30) and (19), leading to

$$v' = h/\alpha H_v \quad , \quad h_v = (1 - \alpha/v)\ H_v \quad , \qquad (67)$$
$$h = h_o \exp \int_o^x (1/\alpha - 1/v)\ dx \ .$$

Noting that $\alpha > 0$, and that $H_v > 0$ by (19), (25), and (57), we conclude that

$$\operatorname{sgn} v' = \operatorname{sgn} h^* = \operatorname{sgn} h_o^* \ . \qquad (68)$$

The Corollary, "$h_o = 0$ implies $h(x) \equiv 0$ and $v \equiv \gamma$" follows immediately. Three types of trajectory are thus distinguished:

a) $v_o < \gamma$, $h(x) < 0$, $v' < 0$, deceleration;

b) $v_o > \gamma$, $h(x) > 0$, $v' > 0$, acceleration;

c) $v_o = \gamma$, $h(x) \equiv 0$, $v \equiv \gamma$, steady state;

In case c) the solution (24) and (31), of the Euler equations, must be replaced by $y \equiv 0$, $v \equiv \gamma$.

13. THE BASIC THEOREMS

Having constructed the solution of the Auxiliary Problem, we shall show that under the assumptions of Theorems 1 or 2 it satisfies the constraints $\psi_2 > 0$ and $\Delta v \geq 0$ on the B-subarc.

Theorem 1: "$2\alpha < \min(a_1, v_1)$ implies: 1) $\alpha < \gamma < 2\alpha$, 2) $\psi_2 > 0$, 3) $\Delta v \geq 0$, with the last two relations holding on the B-subarc of an extremal of the Auxiliary Problem.

To prove 1), observe that from (18.3)

$$\begin{aligned} h(\alpha) &= -f(\alpha), \\ h(2\alpha) &= f(2\alpha)\,[\,\alpha + \tfrac{1}{2}\,k\,(2\alpha)], \end{aligned} \tag{69}$$

and that the hypothesis and (17) imply

$$\begin{aligned} 0 < 2\alpha < a_1 < a_o, \\ k(2\alpha) > 0\,. \end{aligned} \tag{70}$$

Then (69) and $f > 0$ imply $h(\alpha) < 0$ and $h(2\alpha) < 0$. Furthermore, from the hypothesis, $\alpha < 2\alpha < v_1 < v_2$, so that $h = h^*$ on $(\alpha, 2\alpha)$, in view of (58). The conclusion is now implied by the continuity of h^* and by Lemma 1.

To prove 2), observe that ψ_2, defined by (8.1) as

$$\psi_2 \equiv \omega' = y' + v' + 1/\alpha\,, \tag{71}$$

can be exhibited as a function of v, with the aid of (29), (30), (18), (19), in two alternate forms:

$$\begin{aligned} \psi_2(v) &= h/\alpha H_v + f_v/H \\ &= H/\alpha H_v + (f_v/vH_v)[2 + k + (v + k')/(1 + v + k)]\,. \end{aligned} \tag{72}$$

The positiveness of v, f, H, f_v, H_v, $2 + k$, $1 + v + k$ is assured by (16), (18), (19), (20), and (57). There are two possibilities: Either $h \geq 0$ or $h < 0$. If $h \geq 0$, then $\psi_2 > 0$ in the first line of (72). On the other hand, if $h < 0$, then $v < \gamma$ in (65); the previous conclusion, $\gamma < 2\alpha$, and the hypothesis, $2\alpha < a_1$, imply $v < a_1$; then (17) implies $k' > 0$, and finally $\psi_2 > 0$ in the last line of (72).

To prove 3), recall that $v_1 > 2\alpha > \gamma$, and that by Lemma 2, $v > \gamma$ implies $h > 0$, $v' > 0$, and conversely. Then note, with the aid of (41), that $h(v_1) = h(v_2) > 0$. Therefore $v' > 0$ if $v = v_1$ or $v = v_2$, thus excluding the possibility $\Delta v < 0$ in the second line of (46).

For rough practical purposes, the hypothesis of the theorem may be replaced by

$$\kappa M > 4 , \tag{73}$$

where M is the jet Mach number, and κ is the ratio of the atmospheric specific heats. To derive this result observe that: 1) $\alpha = g\ell/c^2 = 1/\kappa M^2$ by the law of perfect gas and the formula for the sonic velocity, 2) the constants a_1 and v_1, which are the values of v at the maximum of k(v) and at the first point of contact of the double tangent to f(v), respectively, lie in the sonic region, 3) the sonic velocity corresponds to $a_0 \sim 1/M$, and both a_1 and v_1 are generally sufficiently near a_0 to justify the inequalities $a_1/a_0 > 1/2$, $v_1/a_0 > 1/2$. Thus, (73) implies $2\alpha < a_1$, and $2\alpha < v_1$.

For the Earth, with $c \sim 2000$ m/s, $g \sim 9.8 m/s^2$, $\ell \sim 8000$ m/s, $\kappa = 1.4$, we calculate

$$\begin{gathered} \alpha \sim 0.02,\ M \sim 6, \\ \kappa M \sim 8.4, \end{gathered} \tag{74}$$

concluding that the hypothesis of the theorem is satisfied for terrestrial rockets.

The vacuum case, $\rho = 0$, solved by Miele, corresponds to $\alpha = 0$ and is, therefore, a subcase of the theorem. On the other hand, the constant-density atmosphere, ρ = const., corresponds to $\alpha = \infty$ and hence lies outside the scope of Theorem 1. Indeed, Leitman succeeded in solving this case only by invoking the quadratic law of drag, C_D = const., bringing the problem within the scope of Theorem 2.

Theorem 2: "If $C_D v^2$ is convex, then $\psi_2 > 0$ and $\Delta v \geq 0$ on the B-subarc of an extremal of the Auxiliary Problem."

In the proof, note that the hypothesis, in view of (15), implies

$$(k + 1)\ (k + 2) + k' > 0, \tag{75}$$

and, consequently, $\psi_2 > 0$ in the last line of (72). Furthermore, observe that (75), (16.4), and (19.2) imply the convexity of $f(v)$. Hence the drag belongs to type 1 of section 7, with no corners on the B-subarc, and with $\Delta v = 0$.

The cases of quadratic law of drag and, the more general, power law of drag, also treated by Miele, appear as subcases of the theorem.

14. SUMMARY

Under the assumptions of Theorems 1 or 2 the solution is characterized by the structural formula

$$(IB)_{N+1}\, C\ ; \qquad (76)$$

i.e., the burning stage B, preceded by an impulsive launching I, contains N additional impulsive thrusts, N being the number of double tangents of the curve $f(v)$, and is followed by the coasting stage without fuel. The solution therefore includes as a special case the results of Tsien and Evans, where $N = 0$. An absolute minimum has been established with the aid of the second variation and a variant of the Sufficiency Principle that is particularly useful in problems of optimum control.

Theorem 1 applies to terrestrial launching and any drag function with some very general properties listed in section 4; Theorem 2 covers extra-terrestrial launching but is restricted to a fairly common class of drags that includes all the cases previously treated in the literature.

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APPENDIX

To prove the Sufficiency Condition stated in section 11, define $w(x,y)$ by

$$
\begin{aligned}
w &\equiv G(x,y) + I^{*} , \\
I^{*} &\equiv \int \left\{ [\lambda.\ g(x,y,u) + \mu\psi]\ dx - \lambda.dy \right\} ,
\end{aligned}
\tag{77}
$$

where I^{*} is the Hilbert integral of the control problem and u, λ, μ belong to the field. With Δ denoting an increment, observe that: 1) $\Delta w = 0$ on a closed path; 2) $\Delta w = 0$ on a boundary subarc in $\Phi = 0$, in virtue of the Transversality Condition (34) and $\mu\psi = 0$; 3) $\Delta w = \Delta G$ on an extremal of the field, in view of $y' = g(x,y,u)$, where $u(x,y)$ is the "slope function".

Next let 0 and 1 denote the end-points of the extremal for $\theta = 0$, and let C_{02} be any admissible arc connecting 0 and a terminal point 2 lying in $\Phi = 0$. Then there follows from the properties of w listed above that

$$
\begin{aligned}
w_o^{\,1} &= G(1) - G(0) , \\
w_1^{\,2} &= 0 \\
w_o^{\,2} &= G(2) - G(0) + I^{*}\ (C_{02}) , \\
w_o^{\,2} &= w_o^1 + w_1^2 ,
\end{aligned}
\tag{78}
$$

leading to

$$
G(1) - G(2) = I^{*}\ (C_{o2}) . \tag{79}
$$

Finally, note that, in view of (53) and $\mu\psi = 0$, the expression for I^{*} in (76) can be also exhibited as

$$
\begin{aligned}
I^{*} &= \int \lambda.\ (g - \bar{g})\ dx = - \int E\ dx , \\
\bar{g} &\equiv g(x,\ y,\ \bar{u}) \quad ; \quad \bar{u} \neq u,
\end{aligned}
\tag{80}
$$

and that II' implies $E > 0$ on C_{02}, and hence $I^{*} < 0$ in (80) and (79). The conclusion

$$
G(1) < G(2) \tag{81}
$$

follows immediately.

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